2.9 Well-Founded Orderings

Literature: Franz Baader and Tobias Nipkow: *Term rewriting and all that*, Cambridge Univ. Press, 1998, Chapter 2.

To show the refutational completeness of resolution, we will make use of the concept of well-founded orderings.

Partial Orderings

A (strict) partial ordering $\succ$ on a set M is a transitive and irreflexive binary relation on M.

An $a \in M$ is called minimal, if there is no b in M such that $a \succ b$.

An $a \in M$ is called smallest, if $b \succ a$ for all $b \in M$ different from a.

Notation

$\prec$ for the inverse relation $\succ^{-1}$

$\succeq$ for the reflexive closure $(\succ \cup =)$ of $\succ$

Well-Foundedness

A (strict) partial ordering $\succ$ is called well-founded (Noetherian), if there is no infinite decreasing chain $a_0 \succ a_1 \succ a_2 \succ \ldots$ with $a_i \in M$.

Well-Founded Orderings: Examples

Natural numbers. $(\mathbb{N}, >)$

Lexicographic orderings. Let $(M_1, \succ_1), (M_2, \succ_2)$ be well-founded orderings. Then let their lexicographic combination

$$\succ \; = (\succ_1, \succ_2)_{lex}$$

on $M_1 \times M_2$ be defined as

$$(a_1, a_2) \succ (b_1, b_2) \;\; :\Leftrightarrow \;\; a_1 \succ_1 b_1, \text{ or else } a_1 = b_1 \;\&\; a_2 \succ_2 b_2$$

(analogously for more than two orderings)

This again yields a well-founded ordering (proof below).

Well-Founded Orderings: Examples

Length-based ordering on words. For alphabets Σ with a well-founded ordering $>_\Sigma$, the relation $\succ$, defined as

$$w \succ w' \quad := \quad \alpha)\ |w| > |w'| \text{ or}$$
$$\beta)\ |w| = |w'| \text{ and } w >_{\Sigma,lex} w',$$

is a well-founded ordering on Σ^* (proof below).

Counterexamples:

$(\mathbb{Z}, >)$;

$(\mathbb{N}, <)$;

the lexicographic ordering on Σ^*

Basic Properties of Well-Founded Orderings

Lemma 2.16:

$(M, \succ)$ is well-founded if and only if every $\emptyset \subset M' \subseteq M$ has a minimal element.

Basic Properties of Well-Founded Orderings

Lemma 2.17:

$(M_i, \succ_i)$ is well-founded for $i = 1, 2$ if and only if $(M_1 \times M_2,\ \succ)$ with $\succ = (\succ_1, \succ_2)_{lex}$ is well-founded.

Proof:

(i) "$\Rightarrow$": Suppose $(M_1 \times M_2,\ \succ)$ is not well-founded. Then there is an infinite sequence $(a_0, b_0) \succ (a_1, b_1) \succ (a_2, b_2) \succ \ldots$.

Let $A = \{a_i \mid i \geq 0\} \subseteq M_1$. Since $(M_1, \succ_1)$ is well-founded, A has a minimal element a_n. But then $B = \{b_i \mid i \geq n\} \subseteq M_2$ can not have a minimal element, contradicting the well-foundedness of $(M_2, \succ_2)$.

(ii) "$\Leftarrow$": obvious.

Noetherian Induction

Theorem 2.18 (Noetherian Induction):
Let $(M, \succ)$ be a well-founded ordering, let Q be a property of elements of M.

If for all $m \in M$ the implication

> if $Q(m')$, for all $m' \in M$ such that $m \succ m'$,[a]
> then $Q(m)$.[b]

is satisfied, then the property $Q(m)$ holds for all $m \in M$.

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[a] induction hypothesis

[b] induction step

Noetherian Induction

Proof:

Let $X = \{m \in M \mid Q(m) \text{ false}\}$. Suppose, $X \neq \emptyset$. Since $(M, \succ)$ is well-founded, X has a minimal element m_1. Hence for all $m' \in M$ with $m' \prec m_1$ the property $Q(m')$ holds. On the other hand, the implication which is presupposed for this theorem holds in particular also for m_1, hence $Q(m_1)$ must be true so that m_1 can not be in X. *Contradiction*.

Multi-Sets

Let M be a set. A multi-set S over M is a mapping $S : M \to \mathbb{N}$. Hereby $S(m)$ specifies the number of occurrences of elements m of the base set M within the multi-set S.

We say that m is an element of S, if $S(m) > 0$.

We use set notation ($\in$, $\subset$, $\subseteq$, $\cup$, $\cap$, etc.) with analogous meaning also for multi-sets, e.g.,

$$
\begin{aligned}
(S_1 \cup S_2)(m) &= S_1(m) + S_2(m) \\
(S_1 \cap S_2)(m) &= \min\{S_1(m), S_2(m)\}
\end{aligned}
$$

Multi-Sets

A multi-set is called finite, if

$$|\{m \in M|\ s(m) > 0\}| < \infty,$$

for each m in M.

From now on we only consider finite multi-sets.

Example. $S = \{a, a, a, b, b\}$ is a multi-set over $\{a, b, c\}$, where $S(a) = 3$, $S(b) = 2$, $S(c) = 0$.

Multi-Set Orderings

Let $(M, \succ)$ be a partial ordering. The multi-set extension of $\succ$ to multi-sets over M is defined by

$$
\begin{aligned}
S_1 \succ_{\text{mul}} S_2 :\Leftrightarrow\ & S_1 \neq S_2 \\
& \text{and } \forall m \in M : [S_2(m) > S_1(m) \\
& \quad \Rightarrow \quad \exists m' \in M : (m' \succ m \text{ and } S_1(m') > S_2(m'))]
\end{aligned}
$$

Theorem 2.19:

a) $\succ_{\text{mul}}$ is a partial ordering.

b) $\succ$ well-founded $\Rightarrow$ $\succ_{\text{mul}}$ well-founded

c) $\succ$ total $\Rightarrow$ $\succ_{\text{mul}}$ total

Proof:

see Baader and Nipkow, page 22–24.